

Black Hole Entropy and the Problem of Universality

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Abstract To derive black hole thermodynamics in any quantum theory of gravity, one must introduce constraints that ensure that a black hole is actually present. For a large class of black holes, the imposition of such “horizon constraints” allows the use of conformal field theory methods to compute the density of states, reproducing the correct Bekenstein-Hawking entropy in a nearly model-independent manner. This approach may explain the “universality” of black hole entropy, the fact that many inequivalent descriptions of quantum states all seem to give the same thermodynamic predictions. It also suggests an elegant picture of the relevant degrees of freedom, as Goldstone-boson-like excitations arising from symmetry breaking by a conformal anomaly induced by the horizon constraints.

1 The Problem of Universality

Nearly thirty-five years have passed since Bekenstein [1] and Hawking [2] first showed us that black holes are thermodynamic objects, with characteristic temperatures

$$T_H = \frac{\hbar \kappa}{2\pi c} \quad (1)$$

and entropies

$$S_{BH} = \frac{A}{4\hbar G} . \quad (2)$$

From the start, it was clear that a statistical mechanical description of these states would be rather peculiar: in contrast to the entropy of an ordinary thermodynamic system, black hole entropy is not extensive, depending on area rather than volume. Moreover, by Wheeler’s famous dictum, “a black hole has no hair”: a classical black

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hole is determined completely by a few macroscopic characteristics, with no apparent room for additional microscopic states. Nevertheless, from the earliest days of black hole thermodynamics, the search for a microscopic understanding has been a vigorous area of research.

Until fairly recently, that search was largely unsuccessful. Some interesting ideas were suggested—entanglement entropy of quantum fields across the horizon [3], or the entropy of quantum fields near the horizon [4]—but these remained speculative. Today, in contrast, a great many physicists can tell you, often in great detail, exactly what microscopic degrees of freedom underlie black hole thermodynamics. The new problem is that they will offer you many *different* explanations. Depending on who you ask, black hole entropy may count

- Weakly coupled string and D-brane states [5, 6]
- Horizonless “fuzzball” geometries [7]
- States in a dual conformal field theory “at infinity” [8, 9]
- Spin network states crossing the horizon [10]
- Spin network states inside the horizon [11]
- Horizon states in a spin foam [12]
- “Heavy” degrees of freedom in induced gravity [13]
- Entanglement entropy [3] (maybe “holographic” [14, 15])
- No local states—it’s inherently global [16]
- Nothing—it comes from quantum field theory in a fixed non-quantum background [2], which knows nothing of quantum gravity
- Maybe something else (points in a causal set in the horizon’s domain of dependence [17]? Kolmogorov-Sinai entropy of strings spreading at the horizon [18]?)

There is, of course, nothing wrong with a healthy competition among candidates for the proper description of the quantum black hole. The relevant degrees of freedom are, after all, presumably quantum gravitational—the Bekenstein-Hawking entropy (2) involves both \hbar and G —and we do not yet have an established quantum theory of gravity. But the fact that so many descriptions give exactly the same answer is a true puzzle.

To see this puzzle more clearly, consider one of the most successful approaches to black hole entropy, that of weakly coupled string theory. To count black hole microstates à la Strominger and Vafa [5], one should proceed as follows:

1. Start with an extremal, supersymmetric, charged black hole;
2. find the horizon area and express it as a function of the charges;
3. “tune down” the gravitational coupling to form a weakly coupled string/brane system;
4. count the states in this weakly coupled system, and express their number in terms of the charges;
5. argue that supersymmetry (or other properties [19]) guarantees that the number of states is the same at strong and weak coupling;
6. compare the results of steps 2 and 4 to determine the entropy as a function of the horizon area.

The method is very effective, even away from extremality, and allows the computation not only of black hole entropy, but of Hawking radiation and even gray-body factors. But the fundamental relationship between entropy and area arises only indirectly, by way of the computation of charges, and this computation is different for each new type of black hole. One cannot use the results from, say, a three-charge black hole in five dimensions to conclude anything about a four-charge black hole in six dimensions, but must recalculate the entropy and horizon area for each new case. Weakly coupled string theory gives the Bekenstein-Hawking entropy, but it gives it one black hole at a time.

2 Conformal Field Theory and the Cardy Formula

The natural question, then, is whether some property of the classical black hole can explain this “universality” by determining the number of quantum states, independent of the details of their description. This is a lot to ask, and I know of only one case in which such a phenomenon occurs. Let us therefore take a brief detour to explore two-dimensional conformal field theory.

A conformal field theory is a field theory that is invariant under both diffeomorphisms (“general covariance”) and Weyl transformations (“local scale invariance” or “conformal invariance”) [20]. In two dimensions, one can always choose complex coordinates; such a theory is then characterized by two symmetry generators $L[\xi]$ and $\bar{L}[\bar{\xi}]$, which generate holomorphic and antiholomorphic diffeomorphisms. The Poisson bracket algebra of these generators is given by the unique central extension of the algebra of two-dimensional diffeomorphisms, the Virasoro algebra:

$$\begin{aligned} \{L[\xi], L[\eta]\} &= L[\eta\xi' - \xi\eta'] + \frac{c}{48\pi} \int dz (\eta'\xi'' - \xi'\eta'') \\ \{\bar{L}[\bar{\xi}], \bar{L}[\bar{\eta}]\} &= \bar{L}[\bar{\eta}\bar{\xi}' - \bar{\xi}\bar{\eta}'] + \frac{\bar{c}}{48\pi} \int d\bar{z} (\bar{\eta}'\bar{\xi}'' - \bar{\xi}'\bar{\eta}'') \\ \{L[\xi], \bar{L}[\bar{\eta}]\} &= 0, \end{aligned} \quad (3)$$

where the central charges c and \bar{c} (the “conformal anomalies”) depend on the particular theory. The zero-mode generators $L_0 = L[\xi_0]$ and $\bar{L}_0 = \bar{L}[\bar{\xi}_0]$ are conserved charges, roughly analogous to energies; their eigenvalues are commonly referred to as “conformal weights” or “conformal dimensions.”

In 1986, Cardy discovered a remarkable property of such theories [21, 22]. Given *any* unitary two-dimensional conformal field theory for which the lowest eigenvalues Δ_0 of L_0 and $\bar{\Delta}_0$ of \bar{L}_0 are nonnegative, the asymptotic density of states at large eigenvalues Δ and $\bar{\Delta}$ takes the form

$$\ln \rho(\Delta, \bar{\Delta}) \sim 2\pi \sqrt{\frac{(c - 24\Delta_0)\Delta}{6}} + 2\pi \sqrt{\frac{(\bar{c} - 24\bar{\Delta}_0)\bar{\Delta}}{6}}, \quad (4)$$

with higher order corrections that are also determined by the symmetry [23, 24, 25]. The entropy is thus fixed by symmetry, independent of any details of the states being counted. Note that upon quantization, after making the usual substitutions $\{\bullet, \bullet\} \rightarrow [\bullet, \bullet]/i\hbar$ and $L_m \rightarrow L_m/\hbar$, a classical central charge c^{cl} contributes c^{cl}/\hbar to the quantum central charge, and a classical conformal “charge” Δ^{cl} contributes Δ^{cl}/\hbar to the quantum conformal weight. The classical piece of a conformal field theory thus yields a term of order $1/\hbar$ in the entropy (4), reproducing the behavior of the Bekenstein-Hawking entropy (2).

At first sight, these results seem irrelevant to our problem. Black holes are not typically two dimensional, and neither are they conformally invariant. There is a sense, though, in which black holes are *nearly* two dimensional and *nearly* conformally invariant near their horizons. Consider, for example, a scalar field ϕ near a black hole horizon. If we write the metric in “tortoise” coordinates

$$ds^2 = N^2(dt^2 - dr_*^2) + ds_\perp^2 \quad \text{with } N \rightarrow 0 \text{ at the horizon,} \quad (5)$$

the Klein-Gordon operator becomes

$$(\square - m^2)\phi = \frac{1}{N^2}(\partial_t^2 - \partial_{r_*}^2)\phi + \mathcal{O}(1), \quad (6)$$

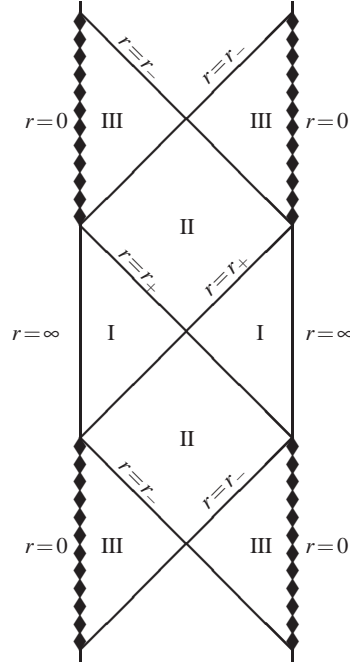
and it is evident that the mass and transverse excitations become negligible as $N \rightarrow 0$. The field is thus effectively described by a two-dimensional conformal field theory [26]. A similar phenomenon occurs for other types of matter, and also, in a sense, for gravity: a generic black hole metric admits an approximate conformal Killing vector near the horizon [27].

Such an effective two-dimensional description has proven very useful in black hole thermodynamics. Building on old results of Chistensen and Fulling [28], Wilczek, Robinson, Iso, Morita, Umetsu, and others have recently shown that the Hawking radiation flux [29] and, indeed, the full thermal spectrum [30, 31] can be extracted from a two-dimensional conformal description, using methods that rely on the conformal anomalies (c, \bar{c}) . Like most derivations of Hawking radiation, these arguments are based on quantum field theory in a fixed black hole background. But as Claudio showed long ago [32], essentially any effective two-dimensional description of gravity also involves a Virasoro algebra, typically with a nonvanishing central charge. We might therefore hope that the conformal description could also tell us about the statistical mechanics of the black hole states themselves.

3 2+1 Dimensions

There is one case in which a conformal field theory derivation of black hole entropy has been completely successful [33, 34]. The Bañados-Teitelboim-Zanelli black hole [35, 36] is a solution of the vacuum Einstein equations in three space-time dimensions with a negative cosmological constant $\Lambda = -1/\ell^2$. Like all vacuum

Fig. 1 The Carter-Penrose diagram for a nonextremal BTZ black hole



spacetimes in 2+1 dimensions, the BTZ geometry has constant curvature, and can be in fact expressed as a quotient of anti-de Sitter space by a discrete group of isometries. Nevertheless, it is a real black hole:

- It has a genuine event horizon at $r = r_+$ and, if the angular momentum is nonzero, an inner Cauchy horizon at $r = r_-$, where r_{\pm} are determined by the mass and angular momentum;
- it occurs as the end point of the gravitational collapse of matter;
- its Carter-Penrose diagram, figure 1, is essentially the same as that of an ordinary Kerr-AdS black hole;
- it exhibits standard black hole thermodynamics, with a temperature and entropy given by (1) and (2), where the horizon “area” is the circumference $A = 2\pi r_+$.

The conformal boundary of a (2+1)-dimensional asymptotically anti-de Sitter spacetime is a two-dimensional cylinder, so it is perhaps not surprising that the algebra of asymptotic symmetries of the BTZ black hole is a Virasoro algebra (3). It is rather more surprising that this algebra has a central extension, but as Brown and Henneaux showed in 1986 [37], the classical central charge, computed from the standard ADM constraint algebra, is nonzero:

$$c = \frac{3\ell}{2G} . \quad (7)$$

The appearance of this central charge can be traced back to the need for boundary terms in the canonical generators of diffeomorphisms, a phenomenon that we under-

stand largely because of the pioneering work of Claudio and his collaborators [38]. Moreover, the classical conformal weights Δ and $\bar{\Delta}$ can be calculated in ordinary canonical general relativity, employing the same methods that are used to determine the ADM mass [37]. Indeed, for the BTZ black hole, the zero modes of the diffeomorphisms are linear combinations of time translations and rotations, and the corresponding conserved quantities are linear combinations of the ordinary ADM mass and angular momentum. A straightforward calculation gives

$$\Delta = \frac{1}{16G\ell}(r_+ + r_-)^2, \quad \bar{\Delta} = \frac{1}{16G\ell}(r_+ - r_-)^2, \quad (8)$$

and the Cardy formula (4) then yields an entropy

$$S = \log \rho \sim \frac{2\pi}{8G}(r_+ + r_-) + \frac{2\pi}{8G}(r_+ - r_-) = \frac{2\pi r_+}{4G}, \quad (9)$$

which may be recognized as precisely the Bekenstein-Hawking entropy.

This derivation is one of the first examples of Maldacena's celebrated AdS/CFT correspondence [8]: the entropy of an asymptotically anti-de Sitter spacetime is determined by the properties of a boundary conformal field theory. It is also a deeply mysterious result. Quantum gravity in three spacetime dimensions has no local degrees of freedom [39], and it is not at all clear where one can find enough degrees of freedom to account for the entropy (9). A review of current proposals can be found in [40]; I will return to this question, in a more general context, in section 6.

The BTZ black hole demonstrates in principle that conformal field theory can be used to compute black hole entropy. Unfortunately, the generalization to higher dimensions is difficult. The derivation of [33, 34] depends crucially on the fact that the conformal boundary of (2+1)-dimensional asymptotically AdS space is a two-dimensional cylinder, which provides a setting for a two-dimensional conformal field theory. No higher-dimensional analog of the Cardy formula is known,¹ so one cannot, at least for now, use symmetries of a higher-dimensional boundary to constrain the density of states.

Moreover, the BTZ computations depend on a symmetry at infinity rather than at the horizon. In 2+1 dimensions this may not matter, since there are no propagating degrees of freedom between the black hole and the conformal boundary, but in higher dimensions, it is less clear how to isolate black hole degrees of freedom. One might argue that a single black hole configuration should make the dominant contribution at infinity, but even this is now known to not always be true [41].

Despite these limitations, the BTZ results have proven surprisingly versatile. In particular, many near-extremal black holes—including most of the black holes whose entropy can be computed using weakly coupled string theory—have a near-horizon geometry of the form $BTZ \times \text{trivial}$, allowing the application of the BTZ method in a more general setting [9]. For generic, nonextremal black holes, though, a more general extension is needed.

¹ Conformal field theory is qualitatively different in two and more than two dimensions: for $d > 2$, the symmetry group has a finite set of generators, but for $d = 2$ it has infinitely many [20].

4 Horizons and Constraints

While the conformal analysis of the BTZ black hole does not extend directly to higher dimensions, it does suggest some interesting directions. We should, perhaps, look for a hidden conformal symmetry, of the type discussed in section 2, with a classical central charge; but we should look near the horizon.

To do so, we must first confront a fundamental conceptual issue. How, in a quantum theory of gravity, do we specify that a black hole is present? In a semiclassical approach, this is easy: we fix a background black hole metric and look at quantum fields and metric fluctuations in that background. In a full quantum theory of gravity, though, we cannot do that: the metric is a quantum operator whose components do not commute, and cannot be simultaneously specified. We must therefore look for a more limited set of constraints that are sufficient to guarantee the presence of the desired black hole while remaining quantum mechanically consistent. The simplest way to do this is to add conditions that ensure the presence of a horizon of some sort—say, an isolated horizon [42]—and study quantum gravity in the presence of these additional constraints. Physically, this amounts to asking questions about conditional probabilities: for instance, “What is the probability of detecting a Hawking radiation photon of energy E , *given* the presence of a horizon of area A ?”

There are several ways to add such “horizon constraints,” which are reviewed in [43]. One approach is to treat the horizon as a sort of boundary. At first sight, this seems a peculiar thing to do: a black hole horizon is certainly not a physical boundary for a freely falling observer. But a horizon *is* a hypersurface at which we can impose “boundary conditions”—namely, the conditions that it is, in fact, a horizon. As in the BTZ case, such restrictions require boundary terms in the generators of diffeomorphisms, whose presence affects their algebra. It can then be shown that in *any* spacetime of dimension greater than two, the subgroup of diffeomorphisms in the r - t plane becomes a Virasoro algebra with the right central charges and conformal weights to yield the Bekenstein-Hawking entropy [44, 45, 46].

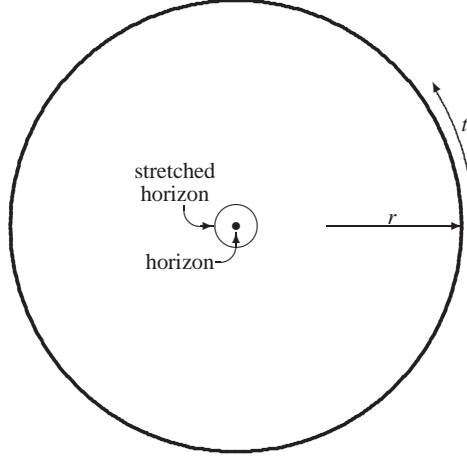
Unfortunately, the diffeomorphisms whose algebra leads to this result are generated by vector fields that blow up at the horizon [47, 48], and this divergence is poorly understood. Moreover, this method does not seem to work for the interesting case of the two-dimensional dilaton black hole. One can therefore look at a slightly different approach, in which the “horizon constraints” are literally imposed as constraints in canonical general relativity [49, 50].

The basic steps of this approach can be summarized as follows:

1. Dimensionally reduce to the “ r - t plane,” which, as argued in section 2, is the relevant setting for near-horizon conformal symmetry. Such a reduction is possible even in the absence of spherical or cylindrical symmetry, although it comes at the expense of an infinite-dimensional Kaluza-Klein gauge group [51]. The action then becomes

$$I = \frac{1}{2} \int d^2x \sqrt{g} \left[\phi R + V[\phi] - \frac{1}{2} W[\phi] h_{IJ} F_{ab}^I F^{Jab} \right], \quad (10)$$

Fig. 2 A “Euclidean” black hole spacetime



where the dilaton ϕ is the dimensionally reduced remnant of the transverse area and F^I is the field strength for the usual Kaluza-Klein gauge field A^I . The potentials V and W depend on the details of the higher-dimensional theory, and need not be further specified.

2. Continue to “Euclidean” signature, as Claudio has often advocated [52]. The metric can then be written in the form

$$ds^2 = N^2 f^2 dr^2 + f^2 (dt + \alpha dr)^2 . \quad (11)$$

For a black hole spacetime, the horizon shrinks to a point and time becomes an angular coordinate (see figure 2), with a period β determined by the geometry.² Rather than evolving in t , we borrow a trick from conformal field theory [20] and evolve radially, starting at a “stretched horizon” just outside $r = 0$.

3. Find the ordinary ADM-style constraints, which take the form

$$\begin{aligned} \mathcal{H}_{\parallel} &= \phi \pi_{\phi} - f \dot{\pi}_f = 0 \\ \mathcal{H}_{\perp} &= f \pi_f \pi_{\phi} + f \left(\frac{\dot{\phi}}{f} \right) - \frac{1}{2} f^2 \hat{V} = 0 \quad \text{with } \hat{V} = V + \frac{h^{IJ} \pi_I \pi_J}{W} \\ \mathcal{H}_I &= \dot{\pi}_I - c^J{}_{IK} A^K \pi_J = 0 . \end{aligned} \quad (12)$$

These can be combined to form Virasoro generators

$$\begin{aligned} L[\xi] &= \frac{1}{2} \int dt \xi (\mathcal{H}_{\parallel} + i \mathcal{H}_{\perp}) \\ \bar{L}[\bar{\xi}] &= \frac{1}{2} \int dt \bar{\xi} (\mathcal{H}_{\parallel} - i \mathcal{H}_{\perp}) , \end{aligned} \quad (13)$$

² Claudio and his collaborators were among the first to study such black holes in two-dimensional dilaton gravity [53].

which satisfy the algebra (3) with vanishing central charge.

4. Determine the geometrical quantities that characterize the black hole:

$$\begin{aligned} \text{the expansion} \quad s &= \varphi \vartheta = f \pi_f - i \dot{\varphi} \\ \text{the surface gravity} \quad \hat{\kappa} &= \pi_\varphi - i \dot{f}/f + f^2 \frac{d\omega}{d\varphi} . \end{aligned} \quad (14)$$

The surface gravity is not unique—in standard general relativity, it depends on the normalization of the Killing vector at the horizon [42], which here appears as a conformal factor ω that will be determined later.

5. Impose horizon constraints to ensure that our initial surface is a stretched horizon. As Claudio noted in [52], the actual horizon is determined by the conditions $s = \bar{s} = 0$. A stretched horizon with surface gravity $\hat{\kappa}_H$ is naturally specified by the slightly loosened conditions

$$\begin{aligned} K &= s - a(\hat{\kappa} - \hat{\kappa}_H) = 0 \\ \bar{K} &= \bar{s} - a(\bar{\hat{\kappa}} - \bar{\hat{\kappa}}_H) = 0 , \end{aligned} \quad (15)$$

where the constant a will be determined below.

6. Note that the horizon constraints K and \bar{K} do not commute with the Virasoro generators (13), which are therefore not symmetries of the constrained system. Cure this problem by using the Bergmann-Komar formulation of Dirac brackets [54]. Let $\{K_i\}$ be a set of constraints for which the inverse Δ_{ij} of $\{K_i, K_j\}$ exists (in Dirac's language, a set of second class constraints). Then for any observable \mathcal{O} , the new observable

$$\mathcal{O}^* = \mathcal{O} - \sum_{i,j} \int du dv \{ \mathcal{O}, K_i(u) \} \Delta_{ij}(u, v) K_j(v) \quad (16)$$

will have vanishing Poisson brackets with the K_i . Since \mathcal{O}^* differs from \mathcal{O} only by a multiple of the constraints K_i , the two are physically equivalent. The Poisson bracket $\{\mathcal{O}_1^*, \mathcal{O}_2^*\}$ can be shown to be equal to the Dirac bracket of \mathcal{O}_1 and \mathcal{O}_2 .

7. Work out the Poisson algebra of the modified Virasoro generators $L^*[\xi]$ and $\bar{L}^*[\bar{\xi}]$. The conformal factor ω in (14) and the constant a in (15) are both fixed by the requirement that these brackets be “nice,” and in particular that they reduce to an ordinary Virasoro algebra at the horizon. Choosing modes

$$\xi_n = \frac{\beta}{2\pi} e^{2\pi i n t / \beta} \quad (17)$$

for the vector fields used to smear the Virasoro generators, we obtain central charges and conformal weights

$$c = \bar{c} = \frac{3\varphi_H}{4G}, \quad \Delta = \bar{\Delta} = \frac{\varphi_H}{16G} \left(\frac{\kappa_H \beta}{2\pi} \right)^2 . \quad (18)$$

8. Use the Cardy formula (4) to obtain an entropy

$$S = \frac{2\pi\varphi_H}{4G} \left(\frac{\kappa_H\beta}{2\pi} \right). \quad (19)$$

This is *almost* the correct Bekenstein-Hawking entropy; it differs from the correct expression by a factor of 2π . I believe this factor has a simple physical explanation: entropy should count the black hole degrees of freedom at a fixed time, but because of our choice of radial evolution, we have computed the entropy at the horizon for *all* times, effectively integrating over a circle of circumference 2π .

5 Universality Again

Now, however, let us recall our original motivation, which was to understand the “universality” of black hole entropy. If the conformal field theory/horizon constraint picture is to explain this universality, it must be the case that the symmetry of the preceding section is secretly present in all of the other derivations of black hole entropy. This is certainly not obvious, but there are a few hopeful signs.

Let us first compare the horizon constraint method to the conformal approach to the BTZ black hole described in section 3. We can start by comparing the central charges and conformal weights:

	<u>BTZ</u>	<u>Horizon CFT</u>	
<i>modes</i>	$\xi_n \sim e^{in(t \pm \ell\phi)/\ell}$	$\xi_n \sim e^{in\kappa_H t}$	
<i>c</i>	$\frac{3\ell}{2G}$	$2\pi \cdot \frac{3\varphi_H}{4\pi G}$	(20)
$\Delta, \bar{\Delta}$	$\frac{(r_+ \pm r_-)^2}{16G\ell}$	$2\pi \cdot \frac{\varphi_H}{32\pi G} \left(\frac{\kappa_H\beta}{2\pi} \right)^2$	

While the entropies agree, it appears that the central charges and conformal weights do not. In fact, though, these disagreements can be traced to two simple sources [50]: the periodicities of the modes do not match, and the BTZ results are based on a coordinate system that is not corotating at the horizon, as one would desire for dimensional reduction. Once these differences are accounted for, the central charges and conformal weights agree precisely. As noted in section 3, the BTZ approach applies also to most of the black holes that can be exactly analyzed with weakly coupled string theory, so this agreement is a significant step.

For loop quantum gravity, the connection is less clear. There is, however, an interesting coincidence that may point toward something deeper. The horizon states of a spin network described in [10] are characterized by a constrained $SL(2, \mathbb{R})$ Chern-Simons theory with coupling constant $k = iA/8\pi\gamma G$, where γ is the Immirzi parameter. Any three-dimensional Chern-Simons theory has an associated two-dimensional conformal field theories, a Wess-Zumino-Witten model that appears, for example,

in the description of boundary states [55]. In the present case, this conformal field theory is Liouville theory, and its central charge is approximately $6k$. If we choose Ashtekar’s original self-dual formulation of loop quantum gravity [56], for which $\gamma = i$, this central charge agrees precisely with the value obtained by the horizon constraint approach.

The central charge (18) also matches that of the “horizon as boundary” approach of [45], and the conformal weights can be obtained as a Komar integral, as suggested in a slightly different context by Emparan and Mateos [57]. I believe it should also be possible to relate this method to the Euclidean path integral approach to black hole entropy; work on this question is in progress.

6 What are the States?

If near-horizon conformal symmetry really provides a universal explanation for black hole statistical mechanics, it had better *not* give us a unique description of the relevant microstates. The problem, after all, is that many different microscopic descriptions seem to yield the same macroscopic thermal properties; picking out one “right” description would miss the point. Nevertheless, it is possible that the derivation of section 4 might give a useful *effective* description of the microscopic degrees of freedom.

Consider the standard Dirac treatment of constraints in quantum mechanics. A set of classical (first class) constraints $L[\xi] = \bar{L}[\bar{\xi}] = 0$ translates to a quantum restriction on the space of states:

$$L[\xi]|\text{phys}\rangle = \bar{L}[\bar{\xi}]|\text{phys}\rangle = 0. \quad (21)$$

But in the presence of a central charge, such a restriction is inconsistent with the Virasoro algebra (3). This is not new, of course, and it is well known how to fix the problem [20]; for example, one can require that only the positive frequency pieces of $L[\xi]$ and $\bar{L}[\bar{\xi}]$ annihilate physical states. The net result, though, is that some states that would have been unphysical in the absence of a central charge must now be considered physical. Equivalently [58], the presence of boundaries or constraints can remove gauge degeneracies among otherwise physically equivalent states, turning “would-be gauge transformations” into new dynamical degrees of freedom.

This phenomenon may have first been observed by Claudio. In an underappreciated passage in [32], he points out that the presence of a central charge in dilaton gravity is quantum mechanically consistent, but results in the appearance of a new degree of freedom. In the present context, we are imposing the constraints (15) only at the horizon, so it is only there that a central charge appears, but the new horizon degree of freedom is essentially the same as Claudio’s.

As Kaloper and Terning have observed [59], this process is also somewhat reminiscent of the Goldstone mechanism, in which a spontaneously broken symmetry gives rise to massless excitations in the direction of the “broken” generators. Here,

of course, the broken symmetry is a gauge symmetry, and the corresponding degrees of freedom are therefore new. But as in the Goldstone mechanism, the pattern of symmetry breaking may give us a universal effective description of the degrees of freedom, while not touching on their “real” structure in terms of the fundamental underlying quantum gravitational states.

For asymptotically anti-de Sitter spacetimes in three dimensions, an explicit description of the symmetry breaking and the corresponding degrees of freedom at infinity is possible [60]. The resulting effective field theory is a Liouville theory. This two-dimensional conformal field theory has the correct central charge and conformal weights to yield the Bekenstein-Hawking entropy via the Cardy formula, but there is still a debate as to whether it really contains enough degrees of freedom [36]. A similar induced action can be found in five-dimensional asymptotically anti-de Sitter gravity [61], although the problem of counting states has not been solved. One might hope for a more general result in arbitrary dimension, perhaps focusing on the horizon rather than infinity; Claudio is responsible for an interesting effort in this direction [52].

One avenue for further research may be to look more carefully at the path integral measure, which is in some sense a count of the number of states, in the presence of a Virasoro algebra with a nonzero central charge. It is known that when second class constraints $\{C_i\}$ are present, the path integral acquires a Fadeev-Popov-like determinant $\det|\{C_i, C_j\}|^{1/2}$ [62]. For a Virasoro algebra, this is

$$D = \det|\{L_m, L_n\}|^{1/2} . \quad (22)$$

A naive evaluation of this expression, using the algebra (3), *almost* gives the Cardy formula: one finds $D \sim \exp\{2\pi\sqrt{6\Delta/c}\}$, which differs from (4) by a flip from $c/6$ to $6/c$. This is a bit too simple, though, since the Virasoro algebra contains an $SL(2, \mathbb{R})$ subgroup, generated by $\{L_0, L_{\pm 1}\}$, whose algebra remains first class. This adds a large degeneracy, increasing the density of states; we really need to evaluate a determinant of the form

$$D = \det \left| -\frac{c}{12} \frac{d^3}{dx^3} + \frac{d}{dx} L + L \frac{d}{dx} \right|^{1/2} \quad \text{with } L = L_0 + L_1 e^{2ix} + L_{-1} e^{-2ix} \quad (23)$$

and trace over appropriate $SL(2, \mathbb{R})$ states. It is not yet clear whether this approach is still too naive; it may be that we need more detailed information about the $SL(2, \mathbb{R})$ representations than can be obtained from a constraint analysis alone.

7 What Next?

While I have given some evidence for the proposal that black hole thermodynamics is effectively determined by near-horizon conformal symmetry, the hypothesis remains very far from being proven. I can think of two main directions to proceed.

First, we should try to connect the near-horizon symmetry more closely to other derivations of black hole entropy. In loop quantum gravity, does the numerical coincidence discussed in section 5 have any deeper significance? Is there a way of using the associated Liouville theory to count states? In the “fuzzball” approach to black holes in string theory [7], no single configuration is expected to have a near-horizon conformal symmetry (or, indeed, a horizon); can a sum over configurations recover such a symmetry? In induced gravity [13], a connection to conformal field theory is already known [63]; can it be tied to the near-horizon symmetry discussed here? Can the determinant (23) be evaluated, and will it give the correct density of states? Does the spin foam method of [12], which relies on a treatment of the horizon as an effective boundary, contain a hidden conformal symmetry?

Second, we should keep in mind that there is more to black hole thermodynamics than the Bekenstein-Hawking entropy. As I noted in section 2, the intensity and spectrum of Hawking radiation can be obtained from an effective two-dimensional theory near the horizon, using conformal field theory methods applied to matter rather than gravity [29, 30, 31]. It is natural to hope that these matter degrees of freedom can be coupled to the near-horizon gravitational degrees of freedom to obtain a dynamical description of Hawking radiation. In 2+1 dimensions, Emparan and Sachs have shown that something of this sort may be possible [64]: a classical scalar field can be coupled to the conformal boundary degrees of freedom of the BTZ black hole, and conformal methods then yield the correct description of Hawking radiation. If this result could be generalized to arbitrary dimensions, with a coupling at the horizon, it would provide very strong evidence for the conformal description of black hole thermodynamics.

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